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The evolution of travelling wave-fronts in a hyperbolic Fisher model. II. The initial-value problem

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Abstract In this paper an initial-value problem for a non-linear hyperbolic Fisher equation is considered in detail. The non-linear hyperbolic Fisher equation is given by

$$\epsilon u_{tt} + u_t = u_{xx} + F(u) + \epsilon F(u)_t,$$

where $\epsilon > 0$ is a parameter and $F(u) = u(1-u)$ is the classical Fisher kinetics. It is established, via the method of matched asymptotic expansions, that the large-time structure of the solution to the initial-value problem involves the evolution of a propagating wavefront which is either of reaction–diffusion or reaction–relaxation type. It is demonstrated that the case $\epsilon = 1$ is a bifurcation point in the sense that for $\epsilon > 1$ the wavefront is of reaction–relaxation type, whereas for $0 < \epsilon < 1$, the wavefront is of reaction–diffusion type.

Keywords Asymptotics · Hyperbolic Fisher equation · Travelling waves

1 Introduction

Parabolic reaction–diffusion equations, such as Fisher's equation, have been extensively studied over recent years. However, these equations arise from the assumption of Fickian diffusion, and have the physically unrealistic property that material diffuses to arbitarily large distances in infinitesimal time. In [1], a remedy to remove this unphysical property, was proposed and discussed in detail. This was based upon a modified Fick's law, which includes a relaxation effect and takes the form (2.3). This modification through relaxation, in the context of thermal diffusion, chemical, population and biological dynamics has been discussed in [2]. The introduction of this modified Fick's law then leads to the hyperbolic Fisher model, which is derived in Sect. 2.

Substantial work on a related hyperbolic Fisher model has been developed by Gallay and Raugel [3–5]. However, the relaxation process included in their model is significantly different to the one presented here, and this leads to significantly differing dynamics once the system moves away from being only weakly hyperbolic, in particular with regard to permanent-form travelling waves.

In a recent paper, [6] (hereafter, referred to as **NK**), the evolution of travelling waves in a weakly hyperbolic generalized Fisher equation of order p (>0) was considered. It was established that the large-time structure of the

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solution involves the evolution of a propagating wave-front. This wave-front is of reaction–diffusion type when $p \geq 1$, while of reaction–relaxation type when $0 < p < 1$.

The results of **NK** (where the relaxation time-scale was considered to be very much smaller than the reaction time-scale) have been extended in [7] (hereafter, referred to as **NL**) by considering the more general situation when the relaxation time-scale is of the same order as, or very much greater than, the chemical time-scale. However, attention was restricted to classical Fisher kinetics (that is, $p = 1$) and consideration of the travelling-wave problem established the conditions under which permanent-form travelling-wave (PTW) solutions exist. Some preliminary numerical solutions of the initial-boundary-value problem where also presented, which illustrate the formation of these wave-fronts.

In this present paper we extend the work of **NL** by using the method of matched asymptotic expansions to obtain the complete large-time structure of the solution to the initial-value problem, with particular emphasis on the propagating wave-front.

2 The hyperbolic Fisher model

We consider the situation in which a chemical species U undergoes simple quadratic autocatalysis in one spatial dimension.

2.1 The system

With $\bar{x}$ measuring distance and t measuring time, the law of mass action then requires

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\int_{\bar{x}_1}^{\bar{x}_2} u\,\mathrm{d}\bar{x}\right\} = [q]_{\bar{x}_1}^{\bar{x}_2} + \int_{\bar{x}_1}^{\bar{x}_2} R(u)\,\mathrm{d}\bar{x}, \quad \bar{x}_2 > \bar{x}_1 \in \mathbb{R}, \quad t > 0. \tag{2.1}$$

Here, u (moles per unit length) is the concentration of the autocatalyst U and q (moles per unit time) is the chemical flux of U. $R(u)$ (moles per unit length per unit time) is the reaction rate, which, for the Fisher kinetics, is given by

$$R(u) = [ku(u_s - u)]^+ = \begin{cases} ku(u_s - u), & u \geq 0, \\ 0, & u < 0 \end{cases} \tag{2.2}$$

with $k > 0$ being the rate constant and $u_s > 0$ the saturation concentration. We observe that $R : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous. To close model (2.1) we must relate q to u. As discussed earlier we do this by modifying the classical Fick's law to include a relaxation effect. Thus we have the modified Fick's law,

$$q + \bar{T} q_t = -D u_{\bar{x}}, \quad \bar{x} \in \mathbb{R}, \quad t > 0. \tag{2.3}$$

Here $D > 0$ is the usual Fickian diffusivity, while $\bar{T} > 0$ is the relaxation time (note that in **NK**, the relaxation time-scale was considered very much smaller than the reaction time-scale). Setting $\bar{T} = 0$ recovers the usual Fick's law.

It is now convenient to introduce dimensionless variables. Using the chemical time-scale and the associated diffusion length-scale, we write

$$t = (ku_s)^{-1} t', \quad \bar{x} = \left(\frac{D}{ku_s}\right)^{\frac{1}{2}} \bar{x}', \quad u = u_s u', \quad q = \left(\frac{ku_s}{D}\right)^{\frac{1}{2}} D u_s q'. \tag{2.4}$$

On substituting from (2.4) in (2.1) and (2.3) (and dropping primes for convenience), we obtain

$$\left.\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t}\left\{\textstyle\int_{\bar{x}_1}^{\bar{x}_2} u\,\mathrm{d}\bar{x}\right\} = [q]_{\bar{x}_1}^{\bar{x}_2} + \int_{\bar{x}_1}^{\bar{x}_2} [u(1-u)]^+\,\mathrm{d}\bar{x}, \quad \bar{x}_2 > \bar{x}_1 \in \mathbb{R}, \quad t > 0. \\ \epsilon q_t + u_{\bar{x}} = -q, \quad \bar{x} \in \mathbb{R}, \quad t > 0. \end{aligned}\right\} \tag{2.5}$$

Here $\epsilon = \bar{T}(ku_s)$ is a dimensionless measure of the ratio of the relaxation time-scale to the chemical reaction time-scale. We now introduce the domain

$$D_T = \{(\bar{x}, t) : -\infty < \bar{x} < \infty, \ 0 < t \leq T\}$$

for any $T > 0$, and to proceed we shall assume, at this stage, that

$$u, q \in BC[\bar{D}_T \setminus \{(0,0)\}], \quad u, q \in C^1[D_T]. \tag{2.6}$$

Under (2.6), Eq. 2.5 may be reduced to differential form, becoming

$$u_t + q_{\bar{x}} = [u(1-u)]^+, \quad \text{on} \ D_T, \tag{2.7a}$$

$$\epsilon q_t + u_{\bar{x}} = -q, \quad \text{on} \ D_T, \tag{2.7b}$$

where $\epsilon > 0$. The initial and boundary conditions to be considered with (2.7) are

$$u(\bar{x}, 0) = \begin{cases} 1, & \bar{x} \leq 0, \\ 0, & \bar{x} > 0, \end{cases} \tag{2.8}$$

$$q(\bar{x}, 0) = 0, \quad -\infty < \bar{x} < \infty, \tag{2.9}$$

$$u(\bar{x}, t), q(\bar{x}, t) \to 0 \quad \text{as} \quad \bar{x} \to \infty, \quad t \in [0, T], \tag{2.10}$$

$$\left.\begin{aligned} u(\bar{x}, t) &\to 1, \\ q(\bar{x}, t) &\to 0, \end{aligned}\right\} \quad \text{as } \bar{x} \to -\infty \quad t \in [0, T] \tag{2.11}$$

We further observe that, with $\epsilon > 0$, the system (2.7) is strictly hyperbolic with wave speeds $\pm\epsilon^{-1/2}$, becoming parabolic when $\epsilon = 0$. Before proceeding further, it is first convenient to rescale Eqs. 2.7. In effect, we rescale time on the relaxation time-scale. We introduce new variables as

$$\bar{x} = \epsilon^{\frac{1}{2}} x, \quad t = \epsilon\tau, \quad u = \bar{u}, \quad q = \epsilon^{-\frac{1}{2}} \bar{q} \tag{2.12}$$

in terms of which (2.7–2.11) become

$$\left.\begin{aligned} \bar{u}_\tau + \bar{q}_x &= \epsilon[\bar{u}(1-\bar{u})]^+, \\ \bar{q}_\tau + \bar{u}_x &= -\bar{q}, \end{aligned}\right\} \quad \text{on} \quad D_T, \tag{2.13}$$

$$\bar{u}(x, 0) = \begin{cases} 1, & x \leq 0, \\ 0, & x > 0, \end{cases} \tag{2.14}$$

$$\bar{q}(x, 0) = 0, \quad -\infty < x < \infty, \tag{2.15}$$

$$\bar{u}(x, \tau), \bar{q}(x, \tau) \to 0 \quad \text{as} \quad x \to \infty, \quad \tau \in [0, T], \tag{2.16}$$

$$\left.\begin{aligned} \bar{u}(x, \tau) &\to 1, \\ \bar{q}(x, \tau) &\to 0, \end{aligned}\right\} \quad \text{as } x \to -\infty \quad \tau \in [0, T]. \tag{2.17}$$

Equations 2.13 can now be written in vector form as

$$\mathbf{v}_\tau + \mathbf{A}\mathbf{v}_x = \mathbf{F}(\mathbf{v}) \quad \text{on} \quad D_T, \tag{2.18}$$

where

$$\mathbf{v} = \begin{pmatrix} \bar{u} \\ \bar{q} \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{F}(\mathbf{v}) = \begin{pmatrix} \epsilon\bar{u}(1-\bar{u}) \\ -\bar{q} \end{pmatrix}. \tag{2.19}$$

The eigenvalues of $\mathbf{A}$ (wave speeds) are now ± 1, confirming that (2.18) is strictly hyperbolic. We next put (2.18) into canonical form. This is achieved by introducing α, β via the invertible linear transformation,

$$\alpha = \frac{1}{2}(\bar{u} + \bar{q}), \quad \beta = \frac{1}{2}(\bar{u} - \bar{q}), \tag{2.20}$$

which has inverse,

$$\bar{u} = \alpha + \beta, \quad \bar{q} = \alpha - \beta. \tag{2.21}$$

In terms of α, β (2.13–2.17) then become

$$\left.\begin{aligned} \alpha_\tau + \alpha_x &= f(\alpha, \beta), \\ \beta_\tau - \beta_x &= g(\alpha, \beta), \end{aligned}\right\} \quad \text{on } D_T, \tag{2.22}$$

$$\alpha(x,0) = \beta(x,0) = \begin{cases} \frac{1}{2}, & x \le 0, \\ 0, & x > 0, \end{cases} \tag{2.23}$$

$$\alpha(x,\tau), \beta(x,\tau) \to \begin{cases} \frac{1}{2}, & x \to -\infty, \\ 0, & x \to \infty, \end{cases} \quad \tau \in [0, T]. \tag{2.24}$$

Here,

$$f(\alpha, \beta) = \begin{cases} \frac{1}{2}\epsilon(\alpha+\beta)(1-\alpha-\beta) - \frac{1}{2}(\alpha - \beta), & \alpha + \beta \ge 0, \\ -\frac{1}{2}(\alpha - \beta), & \alpha + \beta < 0, \end{cases} \tag{2.25}$$

$$g(\alpha, \beta) = \begin{cases} \frac{1}{2}\epsilon(\alpha+\beta)(1-\alpha-\beta) + \frac{1}{2}(\alpha - \beta), & \alpha + \beta \ge 0, \\ \frac{1}{2}(\alpha - \beta), & \alpha + \beta < 0. \end{cases} \tag{2.26}$$

It is instructive now to examine the characteristic form of (2.22). We denote the characteristic families as follows,

$$\left.\begin{aligned} \mathcal{C}_+ &: \quad x = \tau + \text{constant}, \\ \mathcal{C}_- &: \quad x = -\tau + \text{constant} \end{aligned}\right\} \tag{2.27}$$

with $\tau \in [0, T]$. In terms of the two families of characteristic curves, $\mathcal{C}_\pm$, equations (2.22) become

$$\alpha_\tau = f(\alpha, \beta) \quad \text{on } \mathcal{C}_+, \quad \tau \in (0, T], \tag{2.28a}$$

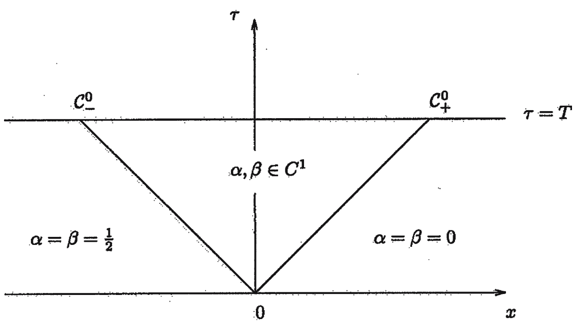
$$\beta_\tau = g(\alpha, \beta) \quad \text{on } \mathcal{C}_-, \quad \tau \in (0, T]. \tag{2.28b}$$

In what follows we divide $\bar{D}_T$ into the following three disjoint domains,

$$\left.\begin{aligned} \text{I} &= \{(x,\tau) \in \bar{D}_T : x > \tau\}, \\ \text{II} &= \{(x,\tau) \in \bar{D}_T : -\tau < x < \tau\}, \\ \text{III} &= \{(x,\tau) \in \bar{D}_T : x < -\tau\}, \end{aligned}\right\} \tag{2.29}$$

and denote the $\mathcal{C}_+$ and $\mathcal{C}_-$ characteristics which eminate from $x = \tau = 0$ as $\mathcal{C}_+^0$ and $\mathcal{C}_-^0$, respectively. An illustration is given in Fig. 1. The initial-value problem (2.22–2.17) has been examined in detail in **NK**, and it is instructive at this point to review their results. In **NK** it was demonstrated that the initial-value problem (2.22–2.17) has no classical solution. However, it does admit a unique, global, weak solution in the sense of regularity according to the characteristic integration of the characteristic equations (2.28) subject to conditions (2.23) and (2.24). This construction allows for simple jump discontinuities in α and β which can only propagate along the characteristics $\mathcal{C}_+$

Fig. 1 The $(x, \tau,)$-plane



and $\mathcal{C}_-$, respectively, together with simple jump discontinuities in the derivatives α_t, α_x, β_t, β_x which are allowed to propagate along either of the $\mathcal{C}_+$ and $\mathcal{C}_-$ characteristics. The construction of this solution must conform everywhere in $\bar{D}_T$ to characteristic integration, according to (2.28) and (2.23), (2.24). From **NK**, this weak solution is global and unique and has,

$$\alpha(x,\tau) = \beta(x,\tau) = 0 \quad \text{in} \quad \text{I}, \tag{2.30}$$

$$\alpha(x,\tau) = \beta(x,\tau) = \frac{1}{2} \quad \text{in} \quad \text{III}. \tag{2.31}$$

Further, it was established in **NK** that the weak solution in domain II must be continuous and continuously differentiable. However, the weak solution must admit discontinuities across the two characteristic curves $\mathcal{C}_+^0$ and $\mathcal{C}_-^0$ in α and β, respectively. In summary, we have from **NK** the following: Concerning $\mathcal{C}_+^0$, we have that

$$[\alpha]_R^L = \alpha_+(\tau) \quad \text{across} \quad \mathcal{C}_+^0. \tag{2.32}$$

Here $[.]_R^L$ indicates the difference between the limit from the left and the limit from the right. Now, from the characteristic equations (2.28a), (2.28b), we have on $\mathcal{C}_+^0$, that $\alpha_+(\tau)$ satisfies,

$$\alpha_+' = f(\alpha_+, 0), \quad \tau \in (0, T], \quad \alpha_+(0) = \frac{1}{2}. \tag{2.33}$$

Here $'$ denotes differentiation with respect to τ. Concerning $\mathcal{C}_-^0$, we have that

$$[\beta]_R^L \equiv \frac{1}{2} - \beta_-(\tau) \quad \text{across} \quad \mathcal{C}_-^0, \tag{2.34}$$

where $\beta_-(\tau)$ satisfies,

$$\beta_-' = g(1/2, \beta_-), \quad \tau \in (0, T], \quad \beta_-(0) = 0. \tag{2.35}$$

Further, we have that

$$\left.\begin{aligned} [\beta_\tau]_R^L &= \tfrac{1}{2} g(\alpha_+(\tau), 0) \\ [\beta_x]_R^L &= -\tfrac{1}{2} g(\alpha_+(\tau), 0) \end{aligned}\right\} \quad \text{across} \quad \mathcal{C}_+^0, \tag{2.36}$$

and

$$\left.\begin{aligned} [\alpha_\tau]_R^L &= -\tfrac{1}{2} f(1/2, \beta_-(\tau)) \\ [\alpha_x]_R^L &= -\tfrac{1}{2} f(1/2, \beta_-(\tau)) \end{aligned}\right\} \quad \text{across} \quad \mathcal{C}_-^0. \tag{2.37}$$

It was further established in **NK** that the weak solution to IVP has,

$$0 \le \alpha(x,\tau) \le \frac{1}{2}, \quad 0 \le \beta(x,\tau) \le \frac{1}{2} \tag{2.38}$$

for all $(x,\tau) \in \bar{D}_\infty$. For the purpose of the present paper it is instructive to re-cast the above in terms of $\bar{u}$ and $\bar{q}$ rather than α and β, which follows via (2.21). We then have,

$$\begin{aligned} &\bar{u}(x,\tau) = \bar{q}(x,\tau) = 0 \quad \text{in I} \\ &\bar{u}(x,\tau) = 1, \ \bar{q}(x,\tau) = 0 \quad \text{in III} \\ &\bar{u}, \bar{q} \in \mathrm{BC}^1 \quad \text{in II} \end{aligned} \tag{2.39}$$

together with the following jumps across $\mathcal{C}_+^0$ and $\mathcal{C}_-^0$,

$$\left.\begin{aligned} [\bar{u}]_R^L &= \alpha_+(\tau), \\ [\bar{q}]_R^L &= \alpha_+(\tau), \end{aligned}\right\} \quad \text{across} \quad \mathcal{C}_+^0.$$

$$\left.\begin{aligned}[\bar{u}]_R^L &= \tfrac{1}{2} - \beta_-(\tau),\\ [\bar{q}]_R^L &= -\tfrac{1}{2} + \beta_-(\tau),\end{aligned}\right\} \quad \text{across} \quad \mathcal{C}_-^0.$$

Further, the problem in domain II is given by

$$\left.\begin{aligned}\bar{u}_\tau + \bar{q}_x &= \epsilon[\bar{u}(1-\bar{u})]^+,\\ \bar{q}_\tau + \bar{u}_x &= -\bar{q},\end{aligned}\right\} \quad -\tau < x < \tau, \quad \tau \in (0, T], \tag{2.40}$$

$$\bar{u}(-\tau,\tau) = \frac{1}{2} + \beta_-(\tau), \quad \bar{q}(-\tau,\tau) = \frac{1}{2} - \beta_-(\tau), \quad \tau \in (0, T], \tag{2.41}$$

$$\bar{u}(\tau,\tau) = \alpha_+(\tau), \quad \bar{q}(\tau,\tau) = \alpha_+(\tau), \quad \tau \in (0, T]. \tag{2.42}$$

It is instructive for what follows to reformulate the problem (2.40–2.42) as a scalar problem.

2.2 The scalar problem

It is the object of this paper to discuss the solution to (2.40–2.42) in domain II, with particular emphasis on the structure of the solution in domain II as $\tau \to \infty$. Since $\bar{u}, \bar{q} \in \mathrm{BC}^1$ in domain II, we can reformulate (2.40–2.42) as a scalar problem, which is convenient for what follows. On applying the operator $\left(\frac{\partial}{\partial\tau} + 1\right)$ to Eq. 2.40_1, and eliminating $\bar{q}$ via Eq. 2.40_2, we arrive at the following scalar problem for $\bar{u} \in \mathrm{BC}^2$ in domain II, namely,

$$\bar{u}_{\tau\tau} + [2\epsilon\bar{u} + (1-\epsilon)]\,\bar{u}_\tau = \bar{u}_{xx} + \epsilon\bar{u}(1-\bar{u}), \quad -\tau < x < \tau, \quad \tau > 0, \tag{2.43}$$

$$\bar{u}(-\tau,\tau) = \frac{1}{2} + \beta_-(\tau), \quad \tau > 0, \tag{2.44}$$

$$\bar{u}(\tau,\tau) = \alpha_+(\tau), \quad \tau > 0, \tag{2.45}$$

where $\alpha_+\colon [0,\infty) \to \mathbb{R}$ and $\beta_- : [0,\infty) \to \mathbb{R}$ are given by the unique, global solution to (2.33) and (2.35), respectively. It is straightforward to solve (2.33) to obtain $\alpha_+(\tau)$, $\tau \geq 0$, explicity as,

$$\alpha_+(\tau) = \begin{cases} \dfrac{(1-\epsilon)\mathrm{e}^{-\frac{(1-\epsilon)}{2}\tau}}{(2-\epsilon)-\epsilon\mathrm{e}^{-\frac{(1-\epsilon)}{2}\tau}}, & 0 < \epsilon < 1,\\ \dfrac{2}{4+\tau}, & \epsilon = 1,\\ \dfrac{(\epsilon-1)}{\epsilon-(2-\epsilon)\mathrm{e}^{-\frac{(\epsilon-1)}{2}\tau}}, & \epsilon > 1. \end{cases} \tag{2.46}$$

We observe immediately that there is a bifurcation in the behaviour of $\alpha_+(\tau)$, as $\tau \to \infty$, when $\epsilon = 1$. Here, specifically,

$$\alpha_+(\tau) \to \begin{cases} 0, & 0 < \epsilon \leq 1\\ 1 - \frac{1}{\epsilon}, & \epsilon > 1 \end{cases}$$

as $\tau \to \infty$. In more detail we have,

$$\alpha_+(\tau) \sim \begin{cases} \left(\frac{1-\epsilon}{2-\epsilon}\right)\mathrm{e}^{-\frac{(1-\epsilon)}{2}\tau} + \ldots, & 0 < \epsilon < 1,\\ \frac{2}{\tau} + \ldots, & \epsilon = 1,\\ 1 - \frac{1}{\epsilon} + \frac{(\epsilon-1)(2-\epsilon)}{\epsilon^2}\mathrm{e}^{-\frac{(\epsilon-1)}{2}\tau} + \ldots, & \epsilon > 1, \end{cases} \tag{2.47}$$

as $\tau \to \infty$. Thus, the discontinuity in $\bar{u}$ across $x = \tau$ decays to zero as $\tau \to \infty$ when $0 < \epsilon \leq 1$ (exponentially in τ when $0 < \epsilon < 1$, but only algebraically in τ when $\epsilon = 1$), but approaches the finite value $\left(1 - \frac{1}{\epsilon}\right)$ as $\tau \to \infty$ (exponentially in τ) when $\epsilon > 1$. To complete the problem (2.43–2.44) we can solve (2.35) to obtain $\beta_-(\tau)$, $\tau \geq 0$, explicitly as,

$$\beta_-(\tau) = \frac{(\epsilon+2)}{2}\,\frac{\left(1 - \mathrm{e}^{-\frac{(1+\epsilon)}{2}\tau}\right)}{(\epsilon+2) + \epsilon\mathrm{e}^{-\frac{(1+\epsilon)}{2}\tau}} \tag{2.48}$$

from which we observe that $\beta_-(\tau) \to \frac{1}{2}$ as $\tau \to \infty$ for each $\epsilon > 0$, with, specifically,

$$\beta_-(\tau) \sim \frac{1}{2} - \left(\frac{\epsilon+1}{\epsilon+2}\right) e^{-\frac{(1+\epsilon)}{2}\tau} + \ldots, \tag{2.49}$$

as $\tau \to \infty$. Thus, for all $\epsilon > 0$, the discontinuity in $\bar{u}$ across $x = -\tau$ decays to zero as $\tau \to \infty$ (exponentially in τ). It is now our purpose to examine the structure of the solution to (2.43–2.45) as $\tau \to \infty$, via the method of matched asymptotic expansions. However, for completeness, we begin with the asymptotic structure of the solution to (2.43–2.45) as $\tau \to 0^+$.

2.3 Asymptotic structure as $\tau \to 0^+$

It follows from (2.46) and (2.48) that

$$\begin{aligned} \alpha_+(\tau) &= \tfrac{1}{2} - \tfrac{1}{8}(2-\epsilon)\tau + O\left(\tau^2\right), \\ \beta_-(\tau) &= \tfrac{1}{8}(2+\epsilon)\tau + O\left(\tau^2\right) \end{aligned} \tag{2.50}$$

as $\tau \to 0^+$. It then follows from (2.44), (2.45), that with $y = \frac{x}{\tau} = O(1)$ as $\tau \to 0^+$, then,

$$\bar{u}(y,\tau) = \frac{1}{2} + \tau f_1(y) + O\left(\tau^2\right) \tag{2.51}$$

as $\tau \to 0^+$ with $-1 \le y \le 1$. On substituting from (2.50) and (2.51) in (2.43–2.45) we find that

$$f_1(y) = -\frac{1}{8}(2y - \epsilon),$$

so that,

$$\bar{u}(y,\tau) = \frac{1}{2} - \frac{1}{8}(2y-\epsilon)\tau + O\left(\tau^2\right)$$

as $\tau \to 0^+$ with $-1 \le y \le 1$. We now consider the structure of the solution to (2.43)–(2.45) as $\tau \to \infty$.

3 Asymptotic structure as $\tau \to \infty$ when $0 < \epsilon < 1$

We first consider the asymptotic solution to (2.43–2.45) as $\tau \to \infty$, for $0 < \epsilon < 1$, via the method of matched asymptotic expansions. In what follows it is natural to again define the scaled coordinate y, via

$$y = \frac{x}{\tau} = O(1), \tag{3.1}$$

as $\tau \to \infty$, where $-1 \le y \le 1$ in domain II.

We begin in region **I** (the sharp hyperbolic wave-front region with propagation speed $\dot{S}_H(\tau) = +1$ and strength $\bar{u} = O\left(e^{-\frac{(1-\epsilon)}{2}\tau}\right)$ as $\tau \to \infty$) where $y = 1 - O\left(\frac{1}{\tau^2}\right)$ as $\tau \to \infty$ [that is, via (3.1), $x = \tau - O\left(\frac{1}{\tau}\right)$ as $\tau \to \infty$]. Thus we introduce the scaled variable $\bar{y}$, where

$$y = 1 + \frac{\bar{y}}{\tau^2} \tag{3.2}$$

with $\bar{y} = O(1)(\le 0)$ as $\tau \to \infty$. We note, via (2.45) and $(2.47)_1$, that in region **I** with $\bar{u} = \bar{u}(\bar{y}, \tau)$, then,

$$\bar{u}(0,\tau) \sim \left(\frac{1-\epsilon}{2-\epsilon}\right) e^{-\frac{(1-\epsilon)}{2}\tau}, \tag{3.3}$$

as $\tau \to \infty$. An examination of (3.3) then leads us to look for for an expansion in region **I** of the form

$$\bar{u}(\bar{y},\tau) = (F(\bar{y}) + o(1))\, e^{-\frac{(1-\epsilon)}{2}\tau}, \tag{3.4}$$

as $\tau \to \infty$ with $\bar{y} = O(1)$ (≤ 0) On substitution of expansion (3.4) in Eq. 2.43 (when written in terms of $\bar{y}$), we obtain the leading-order equation for $F(\bar{y})$ as

$$2\bar{y}F_{\bar{y}\bar{y}} + 2F_{\bar{y}} + \frac{(1+\epsilon)^2}{4}F = 0, \quad \bar{y} < 0. \tag{3.5}$$

Equation 3.5 must be solved subject to satisfying the boundary condition at $\bar{y} = 0$ [that is $x = \tau$] given by (via (3.3) and (3.4))

$$F(0) = \left(\frac{1-\epsilon}{2-\epsilon}\right). \tag{3.6}$$

On writing $\bar{y} = -\frac{2}{(1+\epsilon)^2}s^2$ $(s \geq 0)$, Eq. 3.5 becomes

$$s^2F_{ss} + sF_s - s^2F = 0, \quad s > 0, \tag{3.7}$$

which is the modified Bessel's equation of order zero. Hence the general solution of (3.5) may be written as

$$F(\bar{y}) = A_0\mathrm{I}_0\left[\frac{(1+\epsilon)}{\sqrt{2}}(-\bar{y})^{1/2}\right] + B_0\mathrm{K}_0\left[\frac{(1+\epsilon)}{\sqrt{2}}(-\bar{y})^{1/2}\right], \quad \bar{y} \leq 0, \tag{3.8}$$

where $\mathrm{I}_0[.]$ and $\mathrm{K}_0[.]$ are the usual modified Bessel functions of order zero, and A_0, B_0 are arbitary constants. Applying boundary condition (3.6) (using the small argument asymptotic forms of I_0 and K_0) then requires $A_0 = \left(\frac{1-\epsilon}{2-\epsilon}\right)$, $B_0 = 0$, after which we have

$$F(\bar{y}) = \left(\frac{1-\epsilon}{2-\epsilon}\right)\mathrm{I}_0\left[\frac{(1+\epsilon)}{\sqrt{2}}(-\bar{y})^{1/2}\right], \quad \bar{y} \leq 0. \tag{3.9}$$

Finally, in region **I**, we have

$$\bar{u}(\bar{y}, \tau) = \left(\left(\frac{1-\epsilon}{2-\epsilon}\right)\mathrm{I}_0\left[\frac{(1+\epsilon)}{\sqrt{2}}(-\bar{y})^{1/2}\right] + o(1)\right)\mathrm{e}^{-\frac{(1-\epsilon)}{2}\tau}, \tag{3.10}$$

as $\tau \to \infty$, with $\bar{y} = O(1)$ (≤ 0). We now examine the form of (3.10) for $(-\bar{y}) \gg 1$ (as we move out of region **I** into region **II** where $y = O(1)$ (< 1)) as $\tau \to \infty$. From (3.10) we have,

$$\bar{u}(\bar{y}, \tau) \sim \mathcal{A}(-\bar{y})^{-1/4}\exp\left\{\frac{(1+\epsilon)}{\sqrt{2}}(-\bar{y})^{1/2} - \frac{(1-\epsilon)}{2}\tau\right\} \tag{3.11}$$

with $(-\bar{y}) \gg 1$, where

$$\mathcal{A} = \left(\frac{1-\epsilon}{2-\epsilon}\right)\frac{2^{1/4}}{\sqrt{2\pi}(1+\epsilon)^{1/2}}. \tag{3.12}$$

When written in terms of y, (3.11) becomes,

$$\bar{u}(y, \tau) \sim \exp\left\{\left(\frac{(1+\epsilon)}{\sqrt{2}}(1-y)^{1/2} - \frac{(1-\epsilon)}{2}\right)\tau - \frac{1}{2}\log\tau - \frac{1}{4}\log(1-y) + \log\mathcal{A}\right\}. \tag{3.13}$$

Thus in region **II** we look for an expansion of the form

$$\bar{u}(y, \tau) = \mathrm{e}^{-H(y,\tau)}, \tag{3.14}$$

where

$$H(y, \tau) = h_0(y)\tau + h_1(y)\log\tau + h_2(y) + o(1) \quad \text{as } \tau \to \infty \tag{3.15}$$

with $y = O(1)$ (<1) as $\tau \to \infty$, and $h_0(y) > 0$. On substituting from (3.14) and (3.15) in Eq. 2.43 (when written in terms of y and τ) and solving at each order in turn, we find (after matching with expansion (3.10) of region **I** as $y \to 1^-$) that in region **II** we have

$$\bar{u}(y, \tau) = \exp\left\{\left[\frac{(1+\epsilon)}{2}\left(1-y^2\right)^{\frac{1}{2}} - \frac{(1-\epsilon)}{2}\right]\tau - \frac{1}{2}\log\tau - \hat{H}(y) + o(1)\right\}, \tag{3.16}$$

as $\tau \to \infty$, where $\frac{2\epsilon^{1/2}}{(1+\epsilon)} + o(1) < y < 1 - o(1)$ as $\tau \to \infty$. We note that the function $\hat{H}(y)$ remains undetermined, but matching with region **I** requires that

$$\hat{H}(y) \sim \frac{1}{4}\log(1-y) - \log\mathcal{A} \quad \text{as } y \to 1^-. \tag{3.17}$$

Further, expansion (3.16) becomes non-uniform as $y \to \left(\frac{2\epsilon^{1/2}}{1+\epsilon}\right)^+$ (<1), and in particular when

$$y = \frac{2\epsilon^{1/2}}{(1+\epsilon)} + O\left(\frac{1}{\tau}\right), \tag{3.18}$$

as $\tau \to \infty$. We must therefore introduce a further region, which we denote as region **TW**. Region **TW** is the reaction–diffusion wave-front region. In region **TW**, we have from (3.18) and (3.16), that $\bar{u} = O(1)$ as $\tau \to \infty$. In terms of the original coordinates, we have in this region $x = s(\tau) + z$, where $z = O(1)$ as $\tau \to \infty$, is the travelling-wave coordinate, and

$$s(\tau) = v^*(\epsilon)\tau + \hat{\phi}(\tau) + \phi_0 + o(1) \tag{3.19}$$

as $\tau \to \infty$. Here

$$v^*(\epsilon) = \frac{2\epsilon^{\frac{1}{2}}}{(1+\epsilon)} \tag{3.20}$$

via (3.18), $\hat{\phi}(\tau) = o(\tau)$ as $\tau \to \infty$ is an (as yet) undetermined gauge function (which has $\hat{\phi}(\tau) \to \infty$ as $\tau \to \infty$) and ϕ_0 is a constant. In terms of y, we have in region **TW**,

$$y = \frac{s(\tau)}{\tau} + \frac{z}{\tau}, \tag{3.21}$$

as $\tau \to \infty$, with $z = O(1)$, and via (3.19) $\frac{s(\tau)}{\tau} \sim v^*(\epsilon)$ as $\tau \to \infty$. We now expand in region **TW** in the form

$$\bar{u}(z, \tau) = u_c(z) + o(1) \tag{3.22}$$

as $\tau \to \infty$ with $z = O(1)$. On substituting expansion (3.22) in Eq. 2.43 (when written in terms of z and τ), we obtain the leading-order problem as

$$\left(1 - \left(v^*(\epsilon)\right)^2\right) u_c'' + v^*(\epsilon)[2\epsilon u_c + (1-\epsilon)]u_c' + \epsilon u_c(1-u_c) = 0, \quad -\infty < z < \infty, \tag{3.23}$$

$$u_c(z) > 0, \quad -\infty < z < \infty, \tag{3.24}$$

$$u_c(z) \to 0 \quad \text{as} \quad z \to \infty, \tag{3.25}$$

$$u_c(z) \quad \text{bounded} \quad \text{as } z \to -\infty. \tag{3.26}$$

The condition (3.25) arises from matching expansion (3.22) (as $z \to \infty$) with expansion (3.16) (as $y \to (v^*(\epsilon))^+$). Moreover, a phase-plane analysis of (3.23–3.25) allows boundary condition (3.26) to be replaced by

$$u_c(z) \to 1 \quad \text{as} \quad z \to -\infty. \tag{3.27}$$

The boundary-value problem (3.23–3.27) (for $0 < \epsilon < 1$) has been examined in detail in Sect. 4 of **NL**, where it was established that there exists a unique classical solution to (3.23–3.27), up to translations in z, which we denote as $u_T(z, v^*(\epsilon))$, and represents the minimum speed ($v = v^*(\epsilon)$) permanent form travelling-wave solution. The translational invarience can be absorbed into the constant ϕ_0 in (3.19), and then $u_T(z, v^*(\epsilon))$ is fixed so that $u_T(0, v^*(\epsilon)) = \frac{1}{2}$. Further, we note that from **NL**,

$$u_T(z, v^*(\epsilon)) \sim \begin{cases} \frac{A^*\epsilon^{1/2}}{\sqrt{1-(v^*(\epsilon))^2}} z \exp\left(-\frac{\epsilon^{1/2}z}{\sqrt{1-(v^*(\epsilon))^2}}\right), & z \to +\infty, \\ 1 - \hat{A}\exp\left(\frac{\hat{\lambda}_+\epsilon^{1/2}z}{\sqrt{1-(v^*(\epsilon))^2}}\right), & z \to -\infty, \end{cases} \tag{3.28}$$

where A^*, $\hat{A}$ are fixed positive constants and

$$\hat{\lambda}_+ = -\frac{(1+\epsilon)}{(1-\epsilon)} + \frac{\sqrt{2}}{(1-\epsilon)}\sqrt{1+\epsilon^2} \ \ (>0). \tag{3.29}$$

Hence in region **TW** we have, after determining via balancing arguments in Eq. 2.43 (when written in terms of z and τ), the order of the correction term in expansion (3.22), that

$$\bar{u}(z,\tau) = u_T(z, v^*(\epsilon)) + O\left[\dot{s}(\tau) - v^*(\epsilon)\right] \tag{3.30}$$

with

$$s(\tau) = v^*(\epsilon)\tau + \hat{\phi}(\tau) + \phi_0 + o(1) \quad \text{as} \quad \tau \to \infty. \tag{3.31}$$

It now remains to match (3.30) (as $z \to \infty$) with expansion (3.16) (as $y \to (v^*(\epsilon))^+$). The matching is readily performed (matching $U \equiv \log \bar{u}$ is most convienent) and this determines the gauge function $\hat{\phi}(\tau)$ as

$$\hat{\phi}(\tau) = -\frac{3(1-\epsilon)}{2(1+\epsilon)\epsilon^{1/2}} \log \tau, \tag{3.32}$$

and requires that

$$\hat{H}(y) \sim -\log(y - v^*(\epsilon)) \quad \text{as } y \to (v^*(\epsilon))^+. \tag{3.33}$$

Thus, we have in region **TW** that

$$\bar{u}(z,\tau) = u_T(z, v^*(\epsilon)) + O\left(\tau^{-1}\right) \quad \text{as } \tau \to \infty \tag{3.34}$$

with $z = O(1)$, and

$$s(\tau) = v^*(\epsilon)\tau - \frac{3(1-\epsilon)}{2(1+\epsilon)\epsilon^{1/2}} \log \tau + \phi_0 + o(1) \quad \text{as} \quad \tau \to \infty, \tag{3.35}$$

where $v^*(\epsilon)$ is given by (3.20). Further, we note that the asymptotic speed of this reaction–diffusion wave-front is given by

$$\dot{s}(\tau) = v^*(\epsilon) - \frac{3(1-\epsilon)}{2(1+\epsilon)\epsilon^{1/2}} \frac{1}{\tau} + o\left(\frac{1}{\tau}\right) \tag{3.36}$$

as $\tau \to \infty$. We note that $v^*(\epsilon)$ is the minimum propagation speed identified in **NL**, and that the correction to the propagation speed is algebraic in τ, as $\tau \to \infty$, being $O\left(\tau^{-1}\right)$, as is the rate of convergence to the wave-front.

For $(-z) \gg 1$ we move out of the localized region **TW** into region **III**, where $-1 + o(1) < y < v^*(\epsilon) - o(1)$. On rewriting expansion (3.34) for $(-z) \gg 1$ (obtained from (3.28)(b)), we obtain

$$\bar{u}(z,\tau) \sim 1 - \hat{A} \exp\left(\frac{\hat{\lambda}_+ \epsilon^{\frac{1}{2}} z}{\left[1 - (v^*(\epsilon))^2\right]^{\frac{1}{2}}}\right) \tag{3.37}$$

which in terms of y becomes

$$\bar{u}(y,\tau) \sim 1 - \exp\left(\hat{\lambda}_+ \frac{(1+\epsilon)\epsilon^{1/2}}{(1-\epsilon)}(y - v^*(\epsilon))\tau + \frac{3}{2}\hat{\lambda}_+ \log \tau - \phi_0 \hat{\lambda}_+ \frac{(1+\epsilon)\epsilon^{1/2}}{(1-\epsilon)} + \log \hat{A}\right). \tag{3.38}$$

However, before considering region **III**, it is instructive first to consider the sharp hyperbolic wave-front region with speed $\dot{S}_H(\tau) = -1$ and strength $\bar{u} = 1 - O\left(e^{-\frac{(1+\epsilon)}{2}\tau}\right)$ as $\tau \to \infty$. We label this as region **IV**. The details of this region follow (after minor modifications) those given for region **I** and are summarized here for brevity. To examine region **IV**, where $y = -1 + O\left(\frac{1}{\tau^2}\right)$ as $\tau \to \infty$ we introduce the scaled variable $\hat{y}$, where

$$y = -1 + \frac{\hat{y}}{\tau^2} \tag{3.39}$$

with $\hat{y} = O(1)(\geq 0)$ as $\tau \to \infty$. An examination of (2.44) and (2.48) leads us to look for for an expansion in region **IV** of the form

$$\bar{u}(\hat{y}, \tau) = 1 - G(\hat{y})\mathrm{e}^{-\frac{(1+\epsilon)}{2}\tau} + o\left(\mathrm{e}^{-\frac{(1+\epsilon)}{2}\tau}\right) \tag{3.40}$$

as $\tau \to \infty$ where $G(\hat{y}) = O(1)(>0)$. On substitution of expansion (3.40) in Eq. 2.43 (when written in terms of $\hat{y}$), we obtain the leading-order equation for $G(\hat{y})$ as

$$2\hat{y}G_{\hat{y}\hat{y}} + 2G_{\hat{y}} - \frac{(1-\epsilon)^2}{4}G = 0, \quad \hat{y} > 0. \tag{3.41}$$

Equation 3.41 must be solved subject to satisfying the boundary condition at $\hat{y} = 0$ [that is $x = -\tau$] given by

$$G(0) = \left(\frac{1+\epsilon}{2+\epsilon}\right). \tag{3.42}$$

The solution to (3.41), (3.42) can readily be written down as

$$G(\hat{y}) = \left(\frac{1+\epsilon}{2+\epsilon}\right) \mathrm{I}_0\left[\frac{(1-\epsilon)}{\sqrt{2}}\hat{y}^{1/2}\right], \quad \hat{y} \geq 0, \tag{3.43}$$

where $\mathrm{I}_0[.]$ is the usual modified Bessel function of order zero. Finally, in region **IV**, we have

$$\bar{u}(\hat{y}, \tau) = 1 - \left(\frac{1+\epsilon}{2+\epsilon}\right) \mathrm{I}_0\left[\frac{(1-\epsilon)}{\sqrt{2}}\hat{y}^{1/2}\right]\mathrm{e}^{-\frac{(1+\epsilon)}{2}\tau} + o\left(\mathrm{e}^{-\frac{(1+\epsilon)}{2}\tau}\right), \tag{3.44}$$

as $\tau \to \infty$, with $\hat{y} = O(1)(\geq 0)$. Now as $\hat{y} \to \infty$ we move into region **III**. On consideration of (3.44) we obtain

$$\bar{u}(\hat{y}, \tau) \sim 1 - \left(\frac{1+\epsilon}{2+\epsilon}\right) \frac{2^{1/4}}{\sqrt{2\pi(1-\epsilon)}}\hat{y}^{-1/4} \exp\left(\frac{(1-\epsilon)}{\sqrt{2}}\hat{y}^{1/2} - \frac{(1+\epsilon)}{2}\tau\right) \tag{3.45}$$

as $\tau \to \infty$ with $\hat{y} \gg 1$. On writing (3.45) in terms of y, we obtain

$$\bar{u}(y, \tau) \sim 1 - \mathcal{B}(y+1)^{-1/4}\tau^{-1/2} \exp\left(\frac{(1-\epsilon)}{\sqrt{2}}(y+1)^{1/2} - \frac{(1+\epsilon)}{2}\right)\tau, \tag{3.46}$$

where

$$\mathcal{B} = \left(\frac{1+\epsilon}{2+\epsilon}\right) \frac{2^{1/4}}{\sqrt{2\pi(1-\epsilon)}}.$$

The structure of expansions (3.46) and (3.38) as we move into region **III** from regions **IV** and **TW**, respectively, suggests that in region **III** we write

$$\bar{u}(y, \tau) = 1 - \mathrm{e}^{-\Theta(y,\tau)} \tag{3.47}$$

and expand in the form

$$\Theta(y, \tau) = \Theta_0(y)\tau + \Theta_1(y)\log\tau + \Theta_2(y) + o(1), \tag{3.48}$$

as $\tau \to \infty$ with $-1 + o(1) < y < v^* - o(1)$. It is instructive to consider first the leading-order problem in region **III**. On substituting from (3.47), (3.48) in Eq. 2.43 (when written in terms of y and τ), we obtain at leading order

$$\left(1 - y^2\right)\left(\Theta_0'\right)^2 + y\Theta_0'\left[2\Theta_0 - (1+\epsilon)\right] - (\Theta_0 - 1)(\Theta_0 - \epsilon) = 0 \tag{3.49}$$

with $-1 < y < v^*(\epsilon)$. Equation 3.49 is to be solved subject to matching with region **TW** (as $y \to (v^*(\epsilon))^-$) and with region **IV** (as $y \to -1^+$). The matching condition with region **TW** is given, via (3.38), as

$$\Theta_0(y) \sim -\hat{\lambda}_+\frac{(1+\epsilon)\epsilon^{1/2}}{(1-\epsilon)}\left(y - v^*(\epsilon)\right) \quad \text{as} \quad y \to (v^*(\epsilon))^-. \tag{3.50}$$

Whilst the matching condition with region **IV** requires, via (3.46), that

$$\Theta_0(y) \sim \frac{(1+\epsilon)}{2} - \frac{(1-\epsilon)}{\sqrt{2}}(y+1)^{\frac{1}{2}} \quad \text{as} \quad y \to -1^+. \tag{3.51}$$

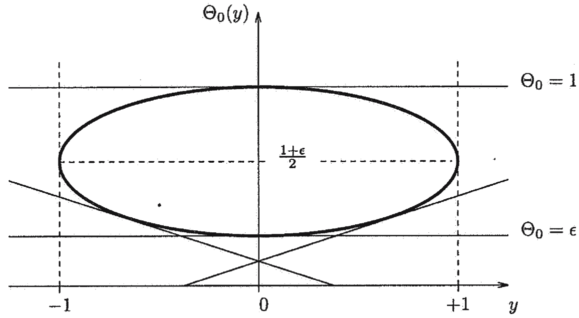


Fig. 2 Schematic representation of the envelope and linear solutions to Eq. 3.49

It is readily established that Eq. 3.49 admits the linear solution

$$\Theta_0(y) = -\hat{\lambda}_+ \frac{(1+\epsilon)\epsilon^{1/2}}{(1-\epsilon)} \left(y - v^*(\epsilon)\right), \quad -1 \le y \le v^*(\epsilon). \tag{3.52}$$

However, expansion (3.47) (with (3.48) and (3.52)) fails to match to expansion (3.44) of region **IV** as $y \to -1^+$. Further consideration of Eq. 3.49 reveals that along with the constant solutions

$$\Theta_0(y) = 1 \quad \text{and} \quad \Theta_0(y) = \epsilon,$$

and the family of linear solutions

$$\Theta_0(y) = \alpha y + \beta \quad \text{where} \quad \alpha^2 = (\beta - 1)(\beta - \epsilon), \quad \beta \in (-\infty, \epsilon] \cup [1, \infty), \tag{3.53}$$

there are also the singular envelope solutions,

$$\Theta_0^{\pm}(y) = \pm \frac{(1-\epsilon)}{2} \left(1 - y^2\right)^{\frac{1}{2}} + \frac{(1+\epsilon)}{2}, \quad -1 \le y \le 1. \tag{3.54}$$

A schematic representation of the solutions to (3.49) is given in Fig. 2. Clearly, on taking

$$\Theta_0(y) = \Theta_0^-(y) = -\frac{(1-\epsilon)}{2}\sqrt{1-y^2} + \frac{(1+\epsilon)}{2}, \tag{3.55}$$

expansion (3.47) (with (3.48)) matches at leading order with expansion (3.44) as $y \to -1$. Hence, we select the following solution in region **III**

$$\Theta_0(y) = \begin{cases} -\frac{(1-\epsilon)}{2}\sqrt{1-y^2} + \frac{(1+\epsilon)}{2}, & -1 < y < y_T, \\ -\hat{\lambda}_+ \frac{(1+\epsilon)\epsilon^{1/2}}{(1-\epsilon)} \left(y - v^*(\epsilon)\right), & y_T \le y < v^*, \end{cases} \tag{3.56}$$

where

$$y_T = \frac{2\gamma}{2\gamma v^*(\epsilon) + (1+\epsilon)}$$

with

$$\gamma = -\hat{\lambda}_+ \frac{(1+\epsilon)\epsilon^{1/2}}{(1-\epsilon)},$$

and we observe that $-1 < y_T < 0$ for $0 < \epsilon < 1$. However, it is important to note that, although $\Theta_0(y)$ and $\Theta_0'(y)$ are continuous for $y \in (-1, v^*(\epsilon))$, the second derivative $\Theta_0''(y)$ is discontinuous at the point $y = y_T$, at which the linear solution (3.52) tangentially meets the singular envelope solution (3.55). This indicates that a thin transition region exists in the neighbourhood of $y = y_T$, in which second-order derivatives are retained a leading order to smooth out this discontinuity in curvature. Hence, to accommodate this transition region, region **III** is replaced by three subregions: region **III(a)** ($y_T + o(1) < y < v^* - o(1)$), region **TR** (transition region, $y = y_T \pm o(1)$) and region **III(b)** ($-1 + o(1) < y < y_T - o(1)$). We consider each of these regions in turn. We begin in region **III(b)**, where $-1 + o(1) < y < y_T - o(1)$. Substitution of (3.47), (3.48) in Eq. 2.43 (when written in terms of y and τ) gives on solving at each order in turn and matching to expansion (3.44) as $y \to -1$, that

$$\bar{u}(y,\tau) = 1 - \exp\left(\left[\frac{(1-\epsilon)}{2}\sqrt{1-y^2} - \frac{(1+\epsilon)}{2}\right]\tau - \frac{1}{2}\log\tau - K(y) + o(1)\right) \tag{3.57}$$

as $\tau \to \infty$ with $-1 + o(1) < y < y_T - o(1)$, where

$$K(y) \sim \frac{1}{4}\log(1+y) - \log\mathcal{B} \tag{3.58}$$

as $y \to -1^+$. The function $K(y)$ remains undetermined. We now consider region **III(a)**. The expansion in region **III(a)** is given by

$$\bar{u}(y,\tau) = 1 - \exp\left(\hat{\lambda}_+\frac{(1+\epsilon)\epsilon^{1/2}}{(1-\epsilon)}\left(y - v^*(\epsilon)\right)\tau - c_1\log\tau - c_1\log|y - y_T| - c_2 + o(1)\right) \tag{3.59}$$

as $\tau \to \infty$, with $y_T + o(1) < y < v^*(\epsilon) - o(1)$ where c_1 and c_2 are constants to be determined on matching with region **TW** as $y \to (v^*(\epsilon))^-$. Matching expansion (3.59) (as $y \to v^*$) with expansion (3.34) (as $z \to -\infty$) requires that

$$c_1 = -\frac{3}{2}\hat{\lambda}_+, \quad c_2 = \frac{3}{2}\hat{\lambda}_+\log(v^* - y_T) - \phi_0\gamma - \log\hat{A}. \tag{3.60}$$

As $y \to y_T^+$ we move into the transition region, region **TR**. An examination of expansion (3.59) (as $y \to y_T$) reveals that in this region $y = y_T + O\left(\tau^{-1/2}\right)$ as $\tau \to \infty$. To examine region **TR** we therefore introduce the scaled coordinate $\eta = (y - y_T)\tau^{1/2}$, where $\eta = O(1)$ as $\tau \to \infty$, and expand as

$$\bar{u}(\eta,\tau) = 1 - \left[h(\eta)\tau^{\kappa} + o(1)\right]\exp\left(\hat{\lambda}_+\frac{(1+\epsilon)\epsilon^{1/2}}{(1-\epsilon)}\left\{\left(y_T - v^*(\epsilon)\right)\tau + \eta\tau^{1/2}\right\}\right) \tag{3.61}$$

as $\tau \to \infty$ with $h(\eta) > 0$ and $\eta = O(1)$, whilst $\kappa = \frac{3}{4}\hat{\lambda}_+$, via (3.59). Substitution of (3.61) in Eq. 2.43 (when written in terms of η and τ) gives at leading order

$$h_{\eta\eta} + \mathcal{C}\frac{1}{2}\eta h_\eta - \kappa\mathcal{C}h = 0, \quad -\infty < \eta < \infty, \tag{3.62}$$

where

$$\mathcal{C} = \frac{(1-\epsilon)}{(1-y_T^2)^{3/2}}. \tag{3.63}$$

On writing $\eta = \mathcal{C}^{-1/2}\xi$, we obtain Eq. 3.62 in the following form

$$h_{\xi\xi} + \frac{\xi}{2}h_\xi - \kappa h = 0, \quad -\infty < \xi < \infty \tag{3.64}$$

with $\kappa > 0$ as given previously. It is established in the appendix that Eq. 3.64 has a solution $h = h^+(\xi)$, where $h^+(\xi)$ is an even function, is strictly positive and has

$$h^+(\xi) \sim |\xi|^{2\kappa} \quad \text{as} \quad |\xi| \to \infty.$$

The general solution to Eq. 3.64 is then given by

$$h(\xi) = C_0 h^+(\xi) + \frac{D_0}{K}h^+(\xi)\int_{-\infty}^{\xi}\frac{\mathrm{e}^{-\frac{s^2}{4}}}{[h^+(s)]^2}\,\mathrm{d}s, \quad -\infty < \xi < \infty, \tag{3.65}$$

where C_0 and D_0 are arbitary constants and

$$K = \int_{-\infty}^{\infty}\frac{\mathrm{e}^{-\frac{s^2}{4}}}{[h^+(s)]^2}\,\mathrm{d}s. \tag{3.66}$$

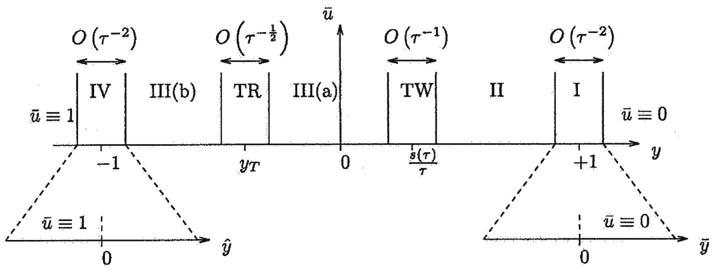


Fig. 3 Schematic representation of the location and thickness of the asymptotic regions as $t \to \infty$ in the subsonic case when $0 < \epsilon < 1$. Note that in this case $y_T = \frac{2\gamma}{2\gamma v^* + (1+\epsilon)}$, where $\gamma = -\hat{\lambda}_+ \frac{(1+\epsilon)\epsilon^{1/2}}{(1-\epsilon)}$. Further, regions I and IV are the sharp hyperbolic wave-front regions, while region TW is the reaction–diffusion wave-front region

Thus, in terms of η, we have

$$h(\eta) = C_0 h^+(\mathcal{C}^{1/2}\eta) + \frac{D_0}{K} h^+(\mathcal{C}^{1/2}\eta) \int_{-\infty}^{\mathcal{C}^{1/2}\eta} \frac{e^{-\frac{s^2}{4}}}{[h^+(s)]^2} \, ds, \quad -\infty < \eta < \infty. \tag{3.67}$$

and we observe that,

$$h(\eta) \sim \begin{cases} (C_0 + D_0)\, \mathcal{C}^\kappa \eta^{2\kappa} & \text{as} \quad \eta \to \infty, \\ C_0 \mathcal{C}^\kappa (-\eta)^{2\kappa} + \frac{2D_0}{K\mathcal{C}^{\kappa+1/2}} (-\eta)^{-(2\kappa+1)} e^{-\frac{\mathcal{C}\eta^2}{4}} & \text{as} \quad \eta \to -\infty. \end{cases} \tag{3.68}$$

We now complete the asymptotic structure in this case by matching expansion (3.61) (with (3.68)) of region **TR** to expansion (3.59) of region **III(a)** (as $\eta \to \infty$) and to expansion (3.57) of region **III(b)** (as $\eta \to -\infty$). Matching requires that

$$C_0 = 0, \quad D_0 = \mathcal{C}^{-\kappa} e^{-c_2} \tag{3.69}$$

and

$$K(y) \sim \left(\frac{3}{2}\hat{\lambda}_+ + 1\right) \log(y_T - y) - \log\left(\frac{2D_0}{K\mathcal{C}^{\kappa+1/2}}\right) \tag{3.70}$$

as $y \to y_T^-$.

This completes the asymptotic structure of the solution to (2.43–2.45) as $\tau \to \infty$ in this case. A schematic representation of the location and thickness of the asymptotic regions as $\tau \to \infty$ is given in Fig. 3. We note that the reaction is totally dormant outside of domain II, with $\bar{u} \equiv 1$ in $y < -1$ [that is, $x < -\tau$] and $\bar{u} \equiv 0$ in $y > 1$ [that is, $x > \tau$]. Regions **I** and **IV** contain the sharp hyperbolic wave-fronts, located at $y = 1$ and $y = -1$, respectively. The hyperbolic wave-front in region **I** has speed $\dot{S}_H(\tau) = +1$ and strength $\bar{u} = O\left(e^{-\frac{(1-\epsilon)}{2}\tau}\right)$ as $\tau \to \infty$, while the hyperbolic wave-front in region **IV** has speed $\dot{S}_H(\tau) = -1$ and strength $(\bar{u} - 1) = O\left(e^{-\frac{(1+\epsilon)}{2}\tau}\right)$ as $\tau \to \infty$. In region **TW** we have the development of a classical reaction–diffusion wave-front of strength $\bar{u} = O(1)$ and speed

$$\dot{S}_{RD}(\tau) = \dot{s}(\tau) = \frac{2\epsilon^{1/2}}{(1+\epsilon)} - \frac{3(1-\epsilon)}{2(1+\epsilon)\epsilon^{1/2}} \frac{1}{\tau} + o\left(\frac{1}{\tau}\right) \quad \text{as} \quad \tau \to \infty. \tag{3.71}$$

Further, we note via (3.34), that the rate of convergence of the solution to (2.43)–(2.45) to the permanent form reaction–diffusion wave is algebraic in τ, being $O(\tau^{-1})$ as $\tau \to \infty$.

4 Asymptotic structure as $\tau \to \infty$ when $\epsilon > 1$

We now consider the asymptotic solution to (2.43–2.45) as $\tau \to \infty$, for $\epsilon > 1$, via the method of matched asymptotic expansions. Again, as in Sect. 3, it is natural to define the scaled coordinate y, via

$$y = \frac{x}{\tau} = O(1), \tag{4.1}$$

as $\tau \to \infty$, where $-1 \le y \le 1$. We begin as in Sect. 3 by anticipating a sharp hyperbolic wave-front region (see region **I** of Sect. 3) with speed $\dot{S}_H = +1$ and via (2.45) and (2.47) strength $\bar{u} = O(1)$. Following Sect. 3 this region is located at $y = 1$ and is of thickness $O\left(\frac{1}{\tau^2}\right)$ as $\tau \to \infty$ [that is, via (4.1), $x = \tau - O\left(\frac{1}{\tau}\right)$ as $\tau \to \infty$]. To examine this region we introduce the scaled variable z where

$$y = 1 + \frac{z}{\tau^2} \tag{4.2}$$

with $z = O(1)(\le 0)$ and look for an expansion of the form

$$\bar{u}(z, \tau) = u_c(z) + o(1), \tag{4.3}$$

as $\tau \to \infty$, where $u_c(z) = O(1)$. However, in this case on substituting expansion (4.3) in Eq. 2.43 (when written in terms of z and τ) no balance can be obtained for a region of thickness $O\left(\tau^{-2}\right)$ as $\tau \to \infty$. On thickening this region we find that a balance can first be achieved when the region is of thickness $O\left(\tau^{-1}\right)$ as $\tau \to \infty$. We denote this region of thickness $O\left(\tau^{-1}\right)$ as region **TW**.

In region **TW**, $y = 1 + O(\tau^{-1})$ as $\tau \to \infty$ [that is $x = \tau + O(1)$ as $\tau \to \infty$]. To examine this region we introduce as before the scaled variable z, where now

$$y = 1 + \frac{z}{\tau} \tag{4.4}$$

with $z = O(1)(\le 0)$ On substituting expansion (4.3) in Eq. 2.43 (when written in terms of z and τ) we obtain the leading-order problem as

$$(\epsilon - 1 - 2\epsilon u)u_c' = \epsilon u(1 - u), \quad -\infty < z < 0, \tag{4.5}$$

$$u_c(z) \ge 0, \tag{4.6}$$

$$u_c(z) \quad \text{bounded as} \quad z \to -\infty, \tag{4.7}$$

$$u_c(z) \to \left(1 - \frac{1}{\epsilon}\right) \quad \text{as} \quad z \to 0^-. \tag{4.8}$$

An examination of Eq. 4.5 requires condition (4.7) to be replaced by

$$u_c(z) \to 1 \quad \text{as} \quad z \to -\infty. \tag{4.9}$$

The boundary-value problem (4.5–4.9) is now recognized as that in **NL** relating to sonic PTW solutions, where it was established that (4.5–4.9) has a unique solution, given by $u_c = u_T(z)$; with $u_T(z)$ having the implicit form,

$$u_T(z)^{\frac{\epsilon-1}{\epsilon+1}}\,(1 - u_T(z)) = \frac{1}{\epsilon}\left(1 - \frac{1}{\epsilon}\right)^{\frac{\epsilon-1}{\epsilon+1}} e^{\frac{\epsilon}{\epsilon+1}z}, \quad z \le 0. \tag{4.10}$$

A sketch of $u_T(z)$ against z, is given in Fig. 4. Further, we note from (4.10), that

$$u_T(z) \sim 1 - \frac{1}{\epsilon}\left(1 - \frac{1}{\epsilon}\right)^{\frac{\epsilon-1}{\epsilon+1}} \exp\left(\frac{\epsilon}{\epsilon+1}z\right) \quad \text{as } z \to -\infty, \tag{4.11}$$

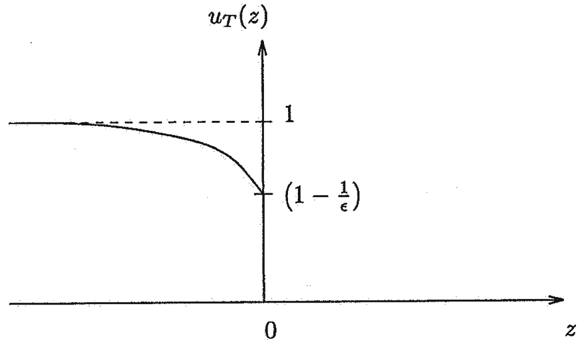
and

$$u_T(z) = \left(1 - \frac{1}{\epsilon}\right) - \frac{1}{\epsilon}z + O\left[z^2\right] \quad \text{as } z \to 0^-. \tag{4.12}$$

For $(-z) \gg 1$ we move out of the localized region **TW** into region **I**, where $-1 + o(1) < y < 1 - o(1)$.

However, before considering region **I** it is instructive, as in Sect. 3, to first consider the sharp hyperbolic wave-front region located at $y = -1$ which has speed $\dot{S}_H(\tau) = -1$ and strength (via (2.45) and (2.47)) $(\bar{u} - 1) = O\left(e^{-\frac{(1+\epsilon)}{2}\tau}\right)$ as $\tau \to \infty$, where $y = -1 + O\left(\tau^{-2}\right)$ as $\tau \to \infty$ [that is, via (4.1), $x = -\tau + O\left(\tau^{-1}\right)$ as $\tau \to \infty$]. The details of

Fig. 4 A sketch of $u_T(z)$ against z, when $\epsilon > 1$



this region, which we denote as region **II**, follow (after minor modifications) those given for region **IV** of Sect. 3 and are summarized here for brevity.

Region II $y = -1 + O\left(\tau^{-2}\right)$ as $\tau \to \infty$

$$\bar{u}(\hat{y}, \tau) = 1 - \left(\frac{\epsilon+1}{\epsilon+2}\right) \mathrm{I}_0\left[\frac{(1-\epsilon)}{\sqrt{2}}\hat{y}^{\frac{1}{2}}\right] \mathrm{e}^{-\frac{(\epsilon+1)}{2}\tau} + o\left(\mathrm{e}^{-\frac{(\epsilon+1)}{2}\tau}\right) \tag{4.13}$$

as $\tau \to \infty$, with $\hat{y} = (y+1)\tau^2 = O(1)(\geq 0)$, and where $\mathrm{I}_0[.]$ is the usual modified Bessel function of order zero. We note that in this case

$$\bar{u}(0, \tau) \sim 1 - \left(\frac{\epsilon+1}{\epsilon+2}\right) \mathrm{e}^{-\frac{(\epsilon+1)}{2}\tau}$$

as $\tau \to \infty$. As $\hat{y} \to \infty$, we move out of the localized region **I** into region **II**, where $-1 + o(1) < y < 1 - o(1)$. We now examine the form of (4.13) for $\hat{y} \gg 1$ (as we move out of region **II** into region **I**). From (4.13) we have,

$$\bar{u}(\hat{y}, \tau) \sim 1 - \mathcal{B}\hat{y}^{-\frac{1}{4}} \exp\left(\frac{(\epsilon-1)}{\sqrt{2}}\hat{y}^{1/2} - \frac{(\epsilon+1)}{2}\tau\right), \tag{4.14}$$

where

$$\mathcal{B} = \left(\frac{1+\epsilon}{2+\epsilon}\right) \frac{2^{1/4}}{\sqrt{2\pi(\epsilon-1)}},$$

as $\tau \to \infty$ with $\hat{y} \gg 1$. On writing (4.14) in terms of y we obtain

$$\bar{u}(y, \tau) \sim 1 - \mathcal{B}(y+1)^{-\frac{1}{4}}\tau^{-\frac{1}{2}} \exp\left(\frac{(\epsilon-1)}{\sqrt{2}}(y+1)^{1/2} - \frac{(\epsilon+1)}{2}\right)\tau \tag{4.15}$$

as $\tau \to \infty$.

We now return to region **I**. The structure of expansions (4.14) and (4.3)(with (4.11) and (4.15)) as we move into region **I** (where $-1 + o(1) < y < 1 - o(1)$) from regions **II** and **TW**, respectively, suggests that in region **I** we write

$$\bar{u}(y, \tau) = 1 - \mathrm{e}^{-\Theta(y,\tau)} \quad \text{as} \quad \tau \to \infty, \tag{4.16}$$

and expand in the form,

$$\Theta(y, \tau) = \Theta_0(y)\tau + \Theta_1(y)\log\tau + \Theta_2(y) + o(1), \tag{4.17}$$

as $\tau \to \infty$ with $-1 + o(1) < y < 1 - o(1)$. It is instructive to consider first the leading-order problem in region **I**. On substituting from (4.16), (4.17) in Eq. 2.43 (when written in terms of y and τ) we obtain at leading order

$$\left(1-y^2\right)\left(\Theta_0'\right)^2 + y\Theta_0'\left[2\Theta_0 - (1+\epsilon)\right] - (\Theta_0 - 1)(\Theta_0 - \epsilon) = 0, \tag{4.18}$$

where $-1 < y < 1$. Equation 4.18 is to be solved subject to matching with region **TW** (as $y \to 1^-$) and with region **II** (as $y \to -1^+$). We note that Eq. 4.18 has already been considered when examining region **III** in Sect. 3. The matching condition with region **TW** is given, via (4.3)(with (4.11)), as

$$\Theta_0(y) \sim \frac{\epsilon}{\epsilon+1}(1-y) \quad \text{as} \quad y \to 1^-. \tag{4.19}$$

Whilst matching to region **II** requires

$$\Theta_0(y) \sim \frac{(1+\epsilon)}{2} + \frac{(1-\epsilon)}{2}\sqrt{1-y^2} \quad \text{as} \quad y \to -1^+. \tag{4.20}$$

It is readily established, via (3.53), that Eq. 4.18 admits the linear solution

$$\Theta_0(y) = \frac{\epsilon}{(\epsilon+1)}(1-y), \quad -1 < y < 1. \tag{4.21}$$

However, expansion (4.16) (with (4.17) and (4.21)) fails to match to expansion (4.14) of region **II** as $y \to -1^+$. Hence, in this case, we must select the singular solution

$$\Theta_0(y) = \Theta_0^+(y) = \frac{(1+\epsilon)}{2} + \frac{(1-\epsilon)}{2}\sqrt{1-y^2}, \tag{4.22}$$

to enable matching with region **I** as $y \to -1^+$. Hence, we must select the following solution in region **I**

$$\Theta_0(y) = \begin{cases} \frac{(1-\epsilon)}{2}\sqrt{1-y^2} + \frac{(1+\epsilon)}{2}, & -1 < y < y_T, \\ \frac{\epsilon}{\epsilon+1}(1-y), & y_T \le y < 1, \end{cases} \tag{4.23}$$

where

$$y_T = -\frac{2\epsilon}{1+\epsilon^2} \; (>-1).$$

However, we note that, although $\Theta_0(y)$ and $\Theta_0'(y)$ are continuous over the range of definitions of (4.23), the second derivative $\Theta_0''(y)$ is discontinuous at the point $y = y_T$, at which the linear solution (4.21) tangentially meets the singular solution (4.22). This indicates that a thin transition region exists in the neighbourhood of $y = y_T$, in which second-order derivatives are retained a leading order to smooth out this discontinuity in curvature.

Hence, to accommodate this transition region, region **I** is replaced by three regions: region **I(b)** ($y_T + o(1) < y < 1 - o(1)$), region **TR** (transition region, $y = y_T \pm o(1)$) and region **I(a)** ($-1 + o(1) < y < y_T - o(1)$). We consider each of these regions in turn. We begin in region in region **I(a)**, where $-1 + o(1) < y < y_T - o(1)$. Substitution of (4.16), (4.17) in Eq. 2.43 (when written in terms of y and τ) gives on solving at each order in turn and matching to expansion (4.14) as $y \to -1$ that

$$\bar{u}(y,\tau) = 1 - \exp\left(\left[\frac{(\epsilon-1)}{2}\sqrt{1-y^2} - \frac{(1+\epsilon)}{2}\right]\tau - \frac{1}{2}\log\tau - \hat{K}(y) + o(1)\right) \tag{4.24}$$

as $\tau \to \infty$ with $-1 + o(1) < y < y_T - o(1)$, where

$$\hat{K}(y) \sim \frac{1}{4}\log(1+y) - \log\mathcal{B} \tag{4.25}$$

as $y \to -1^+$. The function $\hat{K}(y)$ remains undetermined. We now consider region **I(b)**. The expansion in region **I(b)** is given by

$$\bar{u}(y,\tau) = 1 - \exp\left(\frac{\epsilon}{\epsilon+1}(y-1)\tau - c_1\log\tau - c_1\log(y-y_T) - c_2 + o(1)\right) \tag{4.26}$$

as $\tau \to \infty$, with $y_T + o(1) < y < 1 - o(1)$, where c_1 and c_2 are constants to be determined on matching with region **TW** as $y \to 1^-$. Matching expansion (4.26) (as $y \to 1$) with expansion (4.3) (as $z \to -\infty$) requires that

$$c_1 = 0, \quad c_2 = -\log\left(\frac{1}{\epsilon}\left[1 - \frac{1}{\epsilon}\right]^{\frac{\epsilon-1}{\epsilon+1}}\right). \tag{4.27}$$

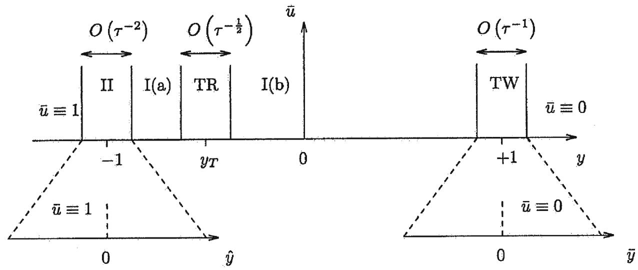


Fig. 5 Schematic representation of the location and thickness of the asymptotic regions as $t \to \infty$ in the subsonic case when $\epsilon > 1$. Note that in this case $y_T = \frac{-2\epsilon}{1+\epsilon^2}$, with $y_T \to -1^+$ as $\epsilon \to 1^+$. Further, region II is the sharp hyperbolic wave-front region, while region TW is the reaction–diffusion wave-front region

As $y \to y_T^+$ we move into the transition region, region **TR**. An examination of expansion (4.26) (as $y \to y_T$) reveals that in this region $y = y_T + O\left(\tau^{-1/2}\right)$ as $\tau \to \infty$. To examine region **TR** we introduce the scaled coordinate $\eta = (y - y_T)\tau^{1/2}$, where $\eta = O(1)$ as $\tau \to \infty$, and expand as

$$\bar{u}(\eta, \tau) = 1 - [F(\eta) + o(1)] \exp\left(-\frac{\epsilon(\epsilon+1)}{1+\epsilon^2}\tau + \frac{\epsilon}{\epsilon+1}\eta\tau^{1/2}\right) \tag{4.28}$$

as $\tau \to \infty$ with $F(\eta) > 0$ and $\eta = O(1)$. Substitution of (4.28) in Eq. 2.43 (when written in terms of η and τ) gives at leading order

$$F_{\eta\eta} + \frac{1}{2}\frac{(\epsilon - 1)}{(1-y_T)^{\frac{3}{2}}}\eta F_\eta = 0, \quad -\infty < \eta < \infty. \tag{4.29}$$

The general solution to (4.29) can readily be obtained as

$$F(\eta) = A_0 \operatorname{erfc}\left(\frac{\eta}{2}\sqrt{\frac{(\epsilon-1)}{(1-y_T)^{\frac{3}{2}}}}\right) + B_0, \tag{4.30}$$

where A_0, B_0 are constants to be determined. Matching expansion (4.28) (as $\eta \to \infty$) with expansion (4.26) (as $y \to y_T^+$) then requires that

$$B_0 = \frac{1}{\epsilon}\left(1 - \frac{1}{\epsilon}\right)^{\frac{\epsilon-1}{\epsilon+1}}. \tag{4.31}$$

Matching expansion (4.28) (as $\eta \to -\infty$) with expansion (4.24) (as $y \to y_T^-$) finally requires that

$$A_0 = -\frac{1}{2\epsilon}\left(1 - \frac{1}{\epsilon}\right)^{\frac{\epsilon-1}{\epsilon+1}} \tag{4.32}$$

with

$$\hat{K}(y) \sim \log|y - y_T| - \mathcal{E} \quad \text{as} \quad y \to y_T^-, \tag{4.33}$$

where

$$\mathcal{E} = -\log\left[\frac{\left(1-\frac{1}{\epsilon}\right)^{\frac{\epsilon-1}{\epsilon+1}}}{\sqrt{2}\,\epsilon\sqrt{\frac{\epsilon-1}{(1-y_T)^{3/2}}}}\right]. \tag{4.34}$$

This completes the asymptotic structure of the solution to (2.43)–(2.45) as $\tau \to \infty$ in this case. A schematic representation of the location and thickness of the asymptotic regions as $t \to \infty$ is given in Fig. 5.

5 Asymptotic structure as $\tau \to \infty$ when $\epsilon = 1$

Finally, we consider the asymptotic solution (2.43–2.45) as $\tau \to \infty$, for $\epsilon = 1$, via the method of matched asymptotic expansions. Again, as in Sects. 3 and 4, it is natural to define the scaled coordinate y, via

$$y = \frac{x}{\tau} = O(1), \tag{5.1}$$

as $\tau \to \infty$, where $-1 \le y \le 1$.

On consideration of the asymptotic structure of the solution (2.43–2.45) as $\tau \to \infty$ when $0 < \epsilon < 1$ (see Sect. 3) and $\epsilon > 1$ (see Sect. 4) we note the following:

(i) $y_T \to -1^+$ and $\frac{s(\tau)}{\tau} \to +1^-$ as $\epsilon \to 1^-$, where y_T and $\frac{s(\tau)}{\tau}$ are the locations of the transition region (region **TR**) and the reaction–diffusion wave-front region (region **TW**), respectively. The locations of regions **TR** and **TW** when $0 < \epsilon < 1$ are illustrated in Fig. 3.
(ii) $y_T \to -1^+$ as $\epsilon \to 1^+$, where y_T is the location of the transition region (region **TR**). The locations of regions **TR** and **TW** when $\epsilon > 1$ are illustrated in Fig. 5.

We further note that the sharp hyperbolic wave-front regions located at $y = -1$ and $y = 1$ encounted in Sect. 3 are not present in this case.

The above indicates that in this case there are two primary asymptotic regions, region **I** ($-1 \le y < 1 - o(1)$) and the reaction–diffusion wave-front region, region **TW**. The details of these regions are for brevity summarized below (with the details following those given for similar regions in earlier sections).

Region I $-1 \le y < 1 - O\left(\tau^{-1}\right)$ as $\tau \to \infty$

The expansion in region **I** is given by

$$\bar{u}(y,\tau) = 1 - \mathrm{e}^{-\frac{(1-y)}{2}\tau} + \frac{\mathrm{e}^{-\tau}}{3} + o\left(\mathrm{e}^{-\tau}\right) \tag{5.2}$$

as $\tau \to \infty$.

Region TW

In region **TW**, $y = 1 + O(\tau^{-1})$ [that is $x = \tau + O(1)$] and $\bar{u} = O(1)$, via boundary condition (2.45) when $\epsilon = 1$ Thus we expand in the form

$$\bar{u}(z,\tau) = u_c(z) + o(1), \tag{5.3}$$

as $\tau \to \infty$ with $z = O(1)$, where $z = (y-1)\tau = (x-\tau)$. On substituting expansion (5.3) in Eq. 2.43 (when written in terms of z and τ) we obtain the leading-order problem as

$$u_c' = -\frac{1}{2}(1-u), \quad -\infty < z < 0, \tag{5.4}$$

$$u_c(z) \ge 0, \tag{5.5}$$

$$u_c(z) \quad \text{bounded as} \quad z \to -\infty, \tag{5.6}$$

$$u_c(z) \to 0 \quad \text{as} \quad z \to 0^-. \tag{5.7}$$

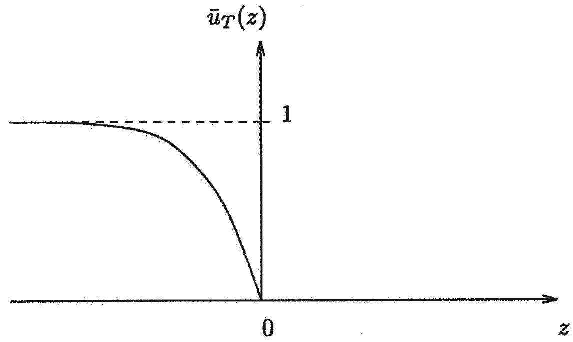
An examination of Eq. 5.4 requires condition (5.6) to be replaced by

$$u_c(z) \to 1 \quad \text{as } z \to -\infty. \tag{5.8}$$

The boundary-value problem (5.4–5.8) is now recognized as that in **NL** relating to sonic PTW solutions, where it was established that (5.4–5.8) has a unique solution, given by $u_c = u_T(z)$, where

$$u_T(z) = \begin{cases} 1 - \mathrm{e}^{\frac{1}{2}z}, & z \le 0, \\ 0, & z > 0, \end{cases} \tag{5.9}$$

Fig. 6 A sketch of $u_T(z)$ against z, when $\epsilon = 1$



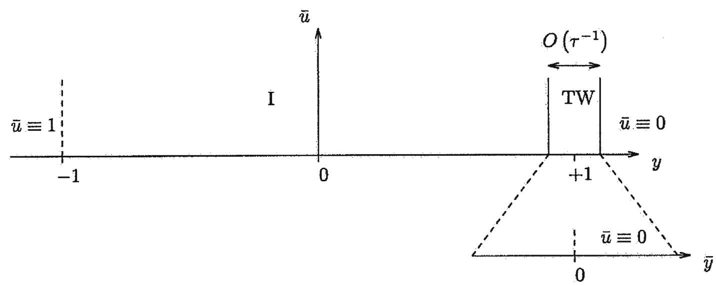


Fig. 7 Schematic representation of the location and thickness of the asymptotic regions as $t \to \infty$ when $\epsilon = 1$. Note that region TW is the reaction–diffusion wave-front region

and translational invariance has been fixed by taking $u_T(0) = 0$. Note that the solution is classical everywhere except at $z = 0$, where there is a gradient discontinuity $[u'_T(z)]_R^L = -\frac{1}{2}$. A sketch of $u_T(z)$ against z is given in Fig. 6.

This completes the asymptotic structure of the solution to (2.43–2.45) as $\tau \to \infty$ in this case. A schematic representation of the location and thickness of the asymptotic regions as $t \to \infty$ is given in Fig. 7.

6 Discussion

The asymptotic structure of the solution, $\bar{u}(x, \tau)$, of (2.43–2.45) in domain II ($-\tau \le x \le \tau$) as $\tau \to \infty$ in the cases $0 < \epsilon < 1$, $\epsilon = 1$ and $\epsilon > 1$ is complete. In each case we have demonstrated, via the method of matched asymptotic expansions, that the large-time structure of the solution to (2.43–2.45) involves the evolution of a propagating travelling-wave front. We note that the reaction is totally dormant outside of domain II, with $\bar{u} \equiv 1$ in domain III ($x < -\tau$) and $\bar{u} \equiv 0$ in domain I ($x > \tau$). In particular, we note that:

(i) $0 < \epsilon < 1$.
In this case we note the presence of sharp hyperbolic wave-fronts located at $x = \pm\tau$. The hyperbolic wave-front in region **I** [located at $x = \tau$] has speed $\dot{S}_H(\tau) = +1$ and strength $\bar{u} = O\left(\mathrm{e}^{-\frac{(1-\epsilon)}{2}\tau}\right)$ as $\tau \to \infty$, while the hyperbolic wave-front in region **IV** [located at $x = -\tau$] has speed $\dot{S}_H(\tau) = -1$ and strength $(\bar{u} - 1) = O\left(\mathrm{e}^{-\frac{(1+\epsilon)}{2}\tau}\right)$ as $\tau \to \infty$. Further, in region **TW** [located at $x = s(\tau)$, where $s(\tau)$ is given by (3.35)] we have the development of a classical reaction–diffusion wave-front of strength $\bar{u} = O(1)$. We have demonstrated that the solution to (2.43–2.45), $\bar{u}(x, \tau)$, has

$$\bar{u}(z, \tau) = u_T(z, v^*(\epsilon)) + O\left(\tau^{-1}\right) \quad \text{as} \quad \tau \to \infty \tag{6.1}$$

with $z = x - s(\tau) = O(1)$, and where $u_T(z, v^*(\epsilon))$ represents the minimum speed permanent form travelling-wave solution. we recall $v^*(\epsilon) = \frac{2\epsilon^{1/2}}{(1+\epsilon)}$ (which is the minimum propagation speed identified in **NL**).

Further, we note that the asymptotic speed of the wave-front is given by

$$\dot{S}_{RD}(\tau) = v^*(\epsilon) - \frac{3(1-\epsilon)}{2(1+\epsilon)\epsilon^{1/2}}\frac{1}{\tau} + o\left(\frac{1}{\tau}\right) \tag{6.2}$$

as $\tau \to \infty$. We note that the correction to the propagation speed is algebraic in τ, as $\tau \to \infty$, being $O\left(\tau^{-1}\right)$, as is the rate of convergence to the wave-front. On rewriting (3.71) in terms of the original variables $\bar{x}$ and t, via (2.12), we obtain

$$\dot{S}_{RD}(t) = \frac{2}{(1+\epsilon)} - \frac{3}{2}\frac{(1-\epsilon)}{(1+\epsilon)}\frac{1}{t} + o\left(\frac{1}{t}\right) \quad \text{as} \quad t \to \infty. \tag{6.3}$$

We note the correspondance of (6.3) to the result of Bramson [8], in the limit as $\epsilon \to 0^+$.

(ii) $\epsilon > 1$.

In this case we note the presence of a sharp hyperbolic wave-front in region **I** (located at $x = -\tau$). The hyperbolic wave-front has speed $\dot{S}_H(\tau) = -1$ and strength $(\bar{u} - 1) = O\left(\mathrm{e}^{-\frac{(1+\epsilon)}{2}\tau}\right)$ as $\tau \to \infty$. However, as in case (iii) when $\epsilon = 1$ the permanent form travelling-wave solution evolves in region **TW** [located at $x = \tau$], and has speed $\dot{s}(\tau) = 1$. We have demonstrated that the solution to (2.43–2.45), $\bar{u}(x, \tau)$, has

$$\bar{u}(z, \tau) = u_T(z, 1) + o(1) \quad \text{as} \quad \tau \to \infty \tag{6.4}$$

with $z = x - \tau = O(1)$, and where $u_T(z, 1)$ represents the weak solution of the travelling wave problem (4.5–4.8) with a jump discontinuity at $z = 0$ [that is, $x = \tau$]. This represents a permanent form reaction–relaxation wave-front

(iii) $\epsilon = 1$.

The permanent form travelling wave solution evolves in region **TW** [located at $x = \tau$], and has speed $\dot{s}(\tau) = 1$. We have demonstrated that the solution to (2.43–2.45), $\bar{u}(x, \tau)$, has

$$\bar{u}(z, \tau) = u_T(z, 1) + o(1) \quad \text{as} \quad \tau \to \infty \tag{6.5}$$

with $z = x - \tau = O(1)$, and where $u_T(z, 1)$ represents the solution to the travelling-wave problem (4.5–4.8). We note that this solution is classical everywhere except at $z = 0$, where there is a gradient discontinuiy $\left[u_T'(z)\right]_R^L = -\frac{1}{2}$. This represents a permanent form hyperbolic reaction–relaxation–diffusion wave-front.

7 Conclusions

In this paper the method of matched asymptotic expansions was used to obtain the complete large-time structure of the solution to an initial-value problem based on the hyperbolic Fisher equation, with particular emphasis on the propagating wavefront. It has been established that the large-time structure of the solution to the initial-value problem involves the evolution of a propagating wavefront which is of reaction–diffusion type or reaction–relaxation type. In particular, when $0 < \epsilon < 1$, we obtain

$$\dot{s}(t) = \frac{2}{(1+\epsilon)} - \frac{3}{2}\frac{(1-\epsilon)}{(1+\epsilon)}\frac{1}{t} + o\left(\frac{1}{t}\right) \tag{7.1}$$

as $t \to \infty$, where $s(t)$ is a measure of the location of the wavefront at time t. This paper has provided the first generalization of the classical result of Bramson (who considered the classical Fisher equation ($\epsilon = 0$) with step initial data) in the context of hyperbolic reaction–diffusion theory. We note the correspondence of (7.1) to the result of Bramson (see [8]), in the limit as $\epsilon \to 0^+$. It was demonstrated that a bifurcation occurs across $\epsilon = 1$, in the following sense: for $0 < \epsilon < 1$ the wavefronts are of classical permanent form reaction–diffusion type with propagation speed $\frac{2}{1+\epsilon}$; for $\epsilon = 1$ the wavefront is non-classical (being a continuous weak solution with a gradient jump) of reaction–relaxation–diffusion type with speed $\epsilon^{-\frac{1}{2}}$; for $\epsilon > 1$ the wavefronts are non-classical (being a weak solution with a jump discontinuity) of reaction–relaxation type with speed $\epsilon^{-\frac{1}{2}}$. We note that when $0 < \epsilon < 1$ ($\epsilon \geq 1$) the wavefront speed is subsonic (sonic) in relation to the hyperbolic wave speed $\epsilon^{-\frac{1}{2}}$, respectively. These results agree with the numerical solutions of the initial-boundary-value problem presented in [7].

Appendix A Analysis of Eq. 3.64

We first consider the initial value problem

$$h'' + \frac{\xi}{2}h' - \kappa h = 0, \quad -\infty < \xi < \infty, \tag{A.1}$$

$$h(0) = 1, \quad h'(0) = 0. \tag{A.2}$$

It is straightforward to establish (via the method of Frobenious) that (A.1), (A.2) has the unique global solution given by $h = \phi_+(\xi)$ where

$$\phi_+(\xi) = 1 + \sum_{n=1}^{\infty} \frac{\kappa!}{(\kappa - n)!(2n)!}\, \xi^{2n}, \quad -\infty < \xi < \infty. \tag{A.3}$$

We observe immediately that $\phi_+(\xi)$ is an even function. Now, a consideration of Eq. A.1 as $|\xi| \to \infty$ establishes that there exists constants $A, B \in \mathbb{R}$, not both zero, such that

$$\phi_+(\xi) \sim A|\xi|^{2\kappa} + B|\xi|^{-(2\kappa+1)}\mathrm{e}^{-\frac{1}{4}\xi^2} \tag{A.4}$$

as $|\xi| \to \infty$. Now suppose that $A = 0$ in (A.4), then $B \neq 0$, and $\phi_+(\xi) \to 0$ as $|\xi| \to \infty$. Thus, via conditions (A.2), there exists a $\xi_0 \in \mathbb{R}$ such that

$$\phi_+(\xi_0) = M = \sup_{\xi \in \mathbb{R}} \phi_+(\xi) > 0, \tag{A.5}$$

$$\phi'_+(\xi_0) = 0, \tag{A.6}$$

$$\phi''_+(\xi_0) \leq 0. \tag{A.7}$$

However, via Eq. A.1, we must have at $\xi = \xi_0$, using (A.5–A.7),

$$\phi''_+(\xi_0) - \kappa M = 0 \tag{A.8}$$

and so, via (A.7),

$$M = \frac{1}{\kappa}\phi''_+(\xi_0) \leq 0$$

which contradicts (A.5). Thus we may conclude that $A \neq 0$. We now set,

$$h_+(\xi) = \frac{1}{A}\phi_+(\xi), \quad -\infty < \xi < \infty. \tag{A.9}$$

It now remains to show that $h_+(\xi) > 0$ for $-\infty < \xi < \infty$. Via (A.9) and (A.4) we have that

$$h_+(\xi) \sim |\xi|^{2\kappa} \quad \text{as} \quad |\xi| \to \infty. \tag{A.10}$$

Now suppose that there exists $\xi_0 \in \mathbb{R}$ such that $h_+(\xi_0) < 0$, then there exists, via (A.10), $\xi_1 \in \mathbb{R}$ such that

$$h_+(\xi_1) = m = \inf_{\xi \in \mathbb{R}} h_+(\xi) < 0, \tag{A.11}$$

$$h'_+(\xi_1) = 0, \tag{A.12}$$

$$h''_+(\xi_1) \geq 0. \tag{A.13}$$

However, via Eq. A.1,

$$h''_+(\xi_1) - \kappa m = 0 \tag{A.14}$$

so that,

$$m = \frac{1}{\kappa}h''_+(\xi_1) \geq 0$$

which contradicts (A.11). We conclude that $h_+(\xi) \geq 0$ for all $\xi \in \mathbb{R}$. Now suppose there exist $\xi_2 \in \mathbb{R}$ such that $h_+(\xi_2) = 0$, then $h'_+(\xi_2) = 0$ and via uniqueness for Eq. A.1,

$$h_+(\xi) = 0 \quad -\infty < \xi < \infty,$$

which conradicts (A.10). We conclude that

$$h_+(\xi) > 0 \quad \text{for all} \quad -\infty < \xi < \infty.$$

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